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Large- N_c Relations Among Isgur-Wise Functions

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Abstract

We investigate the relations that must hold among baryonic Isgur-Wise functions η_i in the large- N_c limit from unitarity constraints, and compare to those found by Chow using the Skyrme model [or $SU(4)$]. Given the exponential dropoff of the η_i away from threshold, unitarity requires only that the usual normalization conditions hold at $w = 1$, and that $\eta = \eta_1$ near threshold. Our results are consistent with, but less powerful than, the Skyrme model relations.

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1 Introduction

QCD simplifies greatly in the chiral [1], heavy-quark [2], and large- N_c [3] limits (where N_c is the number of colors). The Skyrme [4] model of heavy baryons [5, 6, 7] incorporates all of these, and makes powerful predictions [8] about the baryonic Isgur-Wise functions η , η_1 , and η_2 . Since it has been conjectured that all parameter-independent Skyrme model predictions can be derived just from large- N_c unitarity constraints, we investigate what these constraints alone can tell us about Isgur-Wise functions.

Our model-independent large- N_c results turn out to be less powerful than Chow’s Skyrme model predictions. Using the exponential form $\eta \sim \exp\{-\lambda N_c^{3/2}(w-1)\}$ given by Jenkins, Manohar and Wise [7], unitarity requires only that the usual normalization conditions hold at $w = 1$, and that $\eta = \eta_1$ near threshold.

1.1 Baryon Isgur-Wise Functions

The weak transition $\Lambda_b \rightarrow \Lambda_c W$ is characterized by a single Isgur-Wise function, which represents the overlap of the spin-0 light degrees of freedom (“brown muck”):

$$\langle L(v') | L(v) \rangle = \eta(w) \quad (1)$$

where $w \equiv v \cdot v'$. Λ_Q is an isospin singlet ($I=J=0$).

The weak transition $\Sigma_b^{(*)} \rightarrow \Sigma_c^{(*)} W$ is characterized by two other functions, since in this system the brown muck has spin 1:

$$\langle L_\nu(v') | L_\mu(v) \rangle = -\eta_1(w) g_{\mu\nu} + \eta_2(w) v'_\mu v_\nu \quad (2)$$

The Σ_Q and Σ_Q^* can be treated together in the heavy quark limit, as a single “superfield” Σ [9], since they differ only in the relative spin orientation of the heavy quark and the brown muck. Σ is an isospin triplet ($I=J=1$).

The normalization of these functions at $w = 1$ (“threshold”) is:

$$\eta(1) = \eta_1(1) = 1 \quad (3)$$

Heavy quark symmetry makes no prediction for the value of $\eta_2(1)$.

Chow [8] found the following relations among baryon Isgur-Wise functions using the Skyrme model:³

$$\eta_1(w) = -(1+w)\eta_2(w) = \eta(w) \quad (4)$$

These relations are consistent with the normalizations in eq. (3), and additionally predict that $\eta_2(1) = -1/2$.

³Chow writes (ζ_1, ζ_2) for (η_1, η_2) and uses “east coast” metric $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Chow has recently derived the same relations from $SU(4)$ symmetry. [10]

2 Diagrams

2.1 1-Loop Renormalization of $\eta(w)$

In Fig. 1 we show the 1-loop renormalization (vertex and wavefunction) of $\Lambda_b(v) \rightarrow \Lambda_c(v')W$ (*i.e.* of η), which is calculated by Cho [9, eq. (3.4)]. Since $(g_{\Sigma\Lambda}/f)^2 \sim N_c$, the term that it multiplies must vanish at least as fast as $1/N_c$. The relevant piece is

$$\left[3\eta - (2r + w)\eta_1 + (w^2 - 1)\eta_2\right] = \mathcal{O}\left(\frac{1}{N_c}\right), \quad r \equiv \frac{\ln(w + \sqrt{w^2 - 1})}{\sqrt{w^2 - 1}} \quad (5)$$

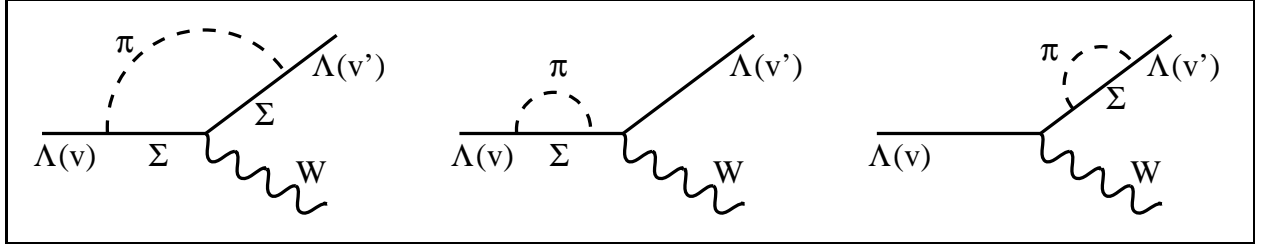


Fig. 1: 1-loop renormalization of $\Lambda_b(v) \rightarrow \Lambda_c(v')W$

One might be tempted to use the renormalization of $\Sigma \rightarrow \Sigma'W$ (*i.e.* of η_1 and η_2), also calculated by Cho, to derive more relations. However, in the large- N_c limit there exists an $I=J$ tower of states above the Λ and Σ . In particular, the state with $I=J=2$ contributes to the 1-loop renormalization of $\Sigma \rightarrow \Sigma'W$. It introduces 3 new Isgur-Wise functions [11, eq. (2.26)], only one of which is normalized at $w = 1$. Thus no useful new information is obtained.

2.2 Single Pion Emission

In Fig. 2, we look at weak decay accompanied by single pion emission: $\Lambda_b(v) \rightarrow \Sigma_c(v')\pi_l(q)W$. The sum of the two diagrams gives an invariant amplitude

$$\mathcal{M} = \frac{g_{\Sigma\Lambda}}{2f} \frac{g_2 V_{cb}}{2\sqrt{2}} \left[\bar{\Sigma}_{ij}^\mu \not{q}^* (1 - \gamma_5) \Lambda \right] \left[\epsilon^{ki} (T_l)_k^j + \epsilon^{kj} (T_l)_k^i \right] (F q_\mu + G v_\mu) \quad (6)$$

where

$$F \equiv \frac{\eta}{v' \cdot q} - \frac{\eta_1}{v \cdot q}, \quad G \equiv \eta_2 \frac{v' \cdot q}{v \cdot q} + \eta_1 - w\eta_2 \quad (7)$$

and the T_l 's are flavor SU(2) generators. We used Cho's [9] Feynman rules restricted to SU(2), so $\{i, j, k\} \in \{1, 2\}$, and $l \in \{1, 2, 3\}$; the group theory factor is just the Clebsch-Gordan coefficient $\langle 1, \alpha; 1, \alpha' | 0, 0 \rangle$. These rules automatically obey unitarity constraints for $\Lambda\pi \rightarrow \Lambda\pi$ analogous to those derived elsewhere [12] for $N\pi \rightarrow N\pi$.

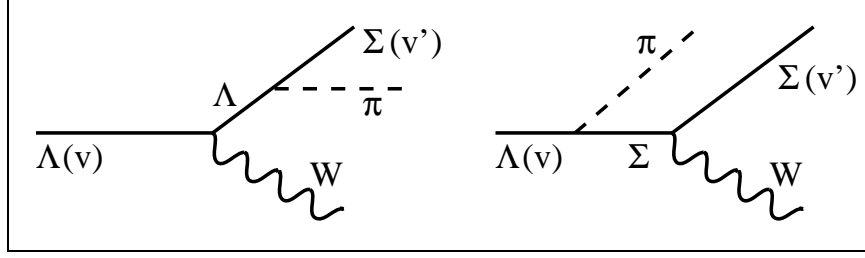


Fig. 2: Single pion emission $\Lambda_b(v) \rightarrow \Sigma_c(v')\pi(q)W$.

Since $(g_{\Sigma\Lambda}/f) \sim \sqrt{N_c}$, the last factor of eq. (6) must vanish at least as fast as $1/\sqrt{N_c}$ when contracted with any final state Σ^μ , which is in turn constrained only by $v'_\mu \Sigma^\mu = 0$:

$$\Sigma^\mu (Fq_\mu + Gv_\mu) = \mathcal{O}\left(\frac{1}{\sqrt{N_c}}\right) \quad (8)$$

for any q satisfying $q^2 = m_\pi^2 \approx 0$ (in the chiral limit) and kinematic constraints.

2.3 Double Pion Emission

Double pion emission, $\Lambda_b(v) \rightarrow \Lambda_c(v')\pi_l(p)\pi_m(q)W$, arises from the 3 diagrams of Fig. 3, plus 3 “crossed” diagrams related by $\{l, p\} \leftrightarrow \{m, q\}$. As long as we restrict our indices to SU(2) as before, the group theory factor of the crossed diagrams equals that of the uncrossed diagrams. Since $(g_{\Sigma\Lambda}/f)^2 \sim N_c$, the remaining term must vanish at least as fast as $1/N_c$:

$$\begin{aligned} 2\eta &+ (\eta_1 - w\eta_2) \left[2w - \frac{v' \cdot p}{v \cdot p} - \frac{v \cdot p}{v' \cdot p} - \frac{v' \cdot q}{v \cdot q} - \frac{v \cdot q}{v' \cdot q} \right] - \eta_2 \left[\frac{(v' \cdot p)(v \cdot q)}{(v' \cdot q)(v \cdot p)} + \frac{(v' \cdot q)(v \cdot p)}{(v' \cdot p)(v \cdot q)} \right] \\ &+ \eta_1 \left[\frac{p \cdot q}{(v' \cdot q)(v \cdot p)} + \frac{p \cdot q}{(v' \cdot p)(v \cdot q)} \right] - \eta \left[\frac{p \cdot q}{(v \cdot p)(v \cdot q)} + \frac{p \cdot q}{(v' \cdot p)(v' \cdot q)} \right] = \mathcal{O}\left(\frac{1}{N_c}\right) \quad (9) \end{aligned}$$

Again, we cannot continue with n -pion emission because higher states in the $I=J$ tower come into play for $n > 2$.

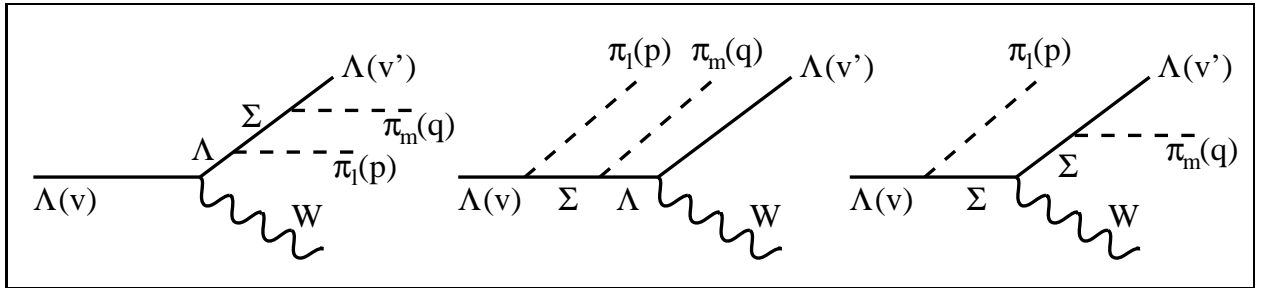


Fig. 3: Two-pion emission, $\Lambda_b(v) \rightarrow \Lambda_c(v')\pi_l(p)\pi_m(q)W$ (3 crossed diagrams not shown).

3 Analysis

3.1 Taylor Expansion

Let $\epsilon^2 = w - 1$. Assume the Isgur-Wise functions can be expanded in ϵ ; then to $\mathcal{O}(\epsilon^2)$,

$$\eta(w) = 1 + \epsilon\eta^{(1)} + \epsilon^2\eta^{(2)}, \quad \eta_1(w) = 1 + \epsilon\eta_1^{(1)} + \epsilon^2\eta_1^{(2)}, \quad \eta_2(w) = \eta_2^{(0)} + \epsilon\eta_2^{(1)} + \epsilon^2\eta_2^{(2)} \quad (10)$$

Eq. (5) becomes

$$\epsilon \left[3\eta^{(1)} - 3\eta_1^{(1)} \right] + \epsilon^2 \left[3\eta^{(2)} - 3\eta_1^{(2)} + 2\eta_2^{(0)} - \frac{1}{3} \right] + \mathcal{O}(\epsilon^3) = \mathcal{O}\left(\frac{1}{N_c}\right) \quad (11)$$

Over different ranges for ϵ , different terms are constrained:

$$\begin{aligned} [\text{With } \epsilon = \mathcal{O}(N_c^{-3/4})] & \quad \eta^{(1)} - \eta_1^{(1)} = \mathcal{O}(N_c^{-1/4}) \\ [\text{With } \epsilon = \mathcal{O}(N_c^{-1/2})] & \quad \eta^{(1)} - \eta_1^{(1)} = \mathcal{O}(N_c^{-1/2}) \\ [\text{With } \epsilon = \mathcal{O}(N_c^{-1/4})] & \quad 3\eta^{(2)} - 3\eta_1^{(2)} + 2\eta_2^{(0)} - \frac{1}{3} = \mathcal{O}(N_c^{-1/4}) \end{aligned} \quad (12)$$

The latter relation is *inconsistent* with Chow's result.

Turning to single-pion emission, we go to the Σ_c rest frame (where $\Sigma^0 = 0$):

$$v' = (1, 0, 0, 0), \quad v = (1 + \epsilon^2, 0, 0, \sqrt{2}\epsilon), \quad q \sim (1, \sin\theta, 0, \cos\theta) \quad (13)$$

and we use the result $\eta^{(1)} - \eta_1^{(1)} = \mathcal{O}(N_c^{-1/2})$ from eq. (12). Then eq. (8) becomes

$$\epsilon (\sqrt{2} \sin\theta) \vec{\Sigma} \cdot (\cos\theta, 0, -\sin\theta) + \mathcal{O}(\epsilon^2) = \mathcal{O}\left(\frac{1}{\sqrt{N_c}}\right) \quad (14)$$

With $\epsilon = \mathcal{O}(N_c^{-1/4})$, this gives a constraint on Σ^μ , representing angular momentum conservation among the light degrees of freedom. We obtain no information about the η 's.

We analyze 2-pion emission in the Λ_c rest frame, with $p \sim (1, \sin\bar{\theta} \cos\bar{\phi}, \sin\bar{\theta} \sin\bar{\phi}, \cos\bar{\theta})$ (the normalization of p and q drop out). Then eq. (9) becomes

$$\begin{aligned} & \epsilon \left[2(1 - p \cdot q)(\eta^{(1)} - \eta_1^{(1)}) \right] \\ & + 2\epsilon^2 \left[\{1 - \cos^2\theta - \cos^2\bar{\theta} + (\eta^{(2)} - \eta_1^{(2)}) + 2\cos\theta \cos\bar{\theta} \eta_2^{(0)}\} \right. \\ & \quad \left. + (p \cdot q) \left\{ -\cos\theta \cos\bar{\theta} - \frac{1}{\sqrt{2}}(\cos\theta + \cos\bar{\theta})(\eta^{(1)} - \eta_1^{(1)}) - (\eta^{(2)} - \eta_1^{(2)}) \right\} \right] \\ & + \mathcal{O}(\epsilon^3) = \mathcal{O}\left(\frac{1}{N_c}\right) \end{aligned} \quad (15)$$

Again using $\eta^{(1)} - \eta_1^{(1)} = \mathcal{O}(N_c^{-1/2})$ from eq. (12), and taking $\epsilon = \mathcal{O}(N_c^{-1/4})$, we find

$$1 - \cos^2\theta - \cos^2\bar{\theta} + (1 - p \cdot q)(\eta^{(2)} - \eta_1^{(2)}) + \cos\theta \cos\bar{\theta} (2\eta_2^{(0)} - p \cdot q) = \mathcal{O}(N_c^{-1/4}) \quad (16)$$

3.2 The Exponential Dropoff

Eq. (16), derived for $\epsilon = \mathcal{O}(N_c^{-1/4})$, cannot be generally true. We conclude that our assumption of analyticity must be invalid this far from threshold. [Nothing significant changes if we try expanding in some other power of $(w - 1)$.] The Isgur-Wise functions must vanish, *e.g.* exponentially, for $\epsilon \geq N_c^{-1/4}$, in which case eq. (9) is trivially satisfied.

Indeed, Jenkins, Manohar and Wise [7] showed that $\eta \sim \exp\{-\lambda N_c^{3/2}(w - 1)\}$. So in fact, the Isgur-Wise functions vanish exponentially fast for $\epsilon > N_c^{-3/4}$, which is a stronger statement than ours.

Unfortunately, the only relation we can then retain is the first line of eq. (12). There is no inconsistency with Chow or with kinematics, but neither can we verify Chow's prediction for $\eta_2(1)$.

4 Conclusions

We have analyzed three weak-decay processes ($\Lambda_b \rightarrow \Lambda_c W$ at one loop, $\Lambda_b \rightarrow \Sigma_c \pi W$, and $\Lambda_b \rightarrow \Lambda_c \pi \pi W$) in the chiral/heavy/large- N_c limits. These are the *only* processes that do not involve higher states in the $I = J$ tower. In this diagrammatic approach, unitarity requires certain constraints on the baryonic Isgur-Wise functions. At threshold, $\eta(1) = \eta_1(1)$ by heavy quark symmetry. For $w - 1 \approx N_c^{-3/2}$, we still find $\eta = \eta_1$, in agreement with Chow. The functions vanish exponentially beyond that, and we can derive no further information.

These unitarity constraints are consistent with, but not as powerful as, Chow's Skyrme model [or $SU(4)$] relations [8, 10]. In particular, unitarity constraints give no prediction for $\eta_2(1)$, whereas the Skyrme model analysis predicts $\eta_2(1) = -1/2$.

We emphasize that we do *not* disagree with Chow's results. Rather, we have shown that perturbative unitarity is incapable of reproducing them. Here is an explicit counterexample to the widely-held belief that all large- N_c predictions can be derived from unitarity constraints.

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